



ON CONTACT PROBLEMS FOR A MEDIUM WITH RIGID FLAT INCLUSIONS OF ARBITRARY SHAPE

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Abstract—Consider the elasto-static problem in which the contact between two compressed dissimilar elastic half-spaces is perturbed by disc inclusions of various constant thicknesses, but arbitrary shapes. We show that the interfacial normal displacement is zero where the half-spaces are in contact, and present some generalizations of the result.

1. INTRODUCTION

In a recent paper Selvadurai (1993) considered the problem of a rigid disc inclusion of constant thickness (i.e. a penny) compressed between dissimilar isotropic elastic half-spaces, as shown in Fig. 1. He formulated this axisymmetric problem in terms of Hankel transforms and found, during the course of extensive manipulation, that

(1) the displacements Δ_1, Δ_2 are related by

$$\mathfrak{G}_1 \Delta_1 = \mathfrak{G}_2 \Delta_2 \quad (1)$$

where $\mathfrak{G}_i = \mu_i / (1 - \nu_i)$;

- (2) the contact stresses on either side of the inclusion are identical;
 (3) the normal displacements in the region where the half-spaces are in contact are not only equal, but are separately zero.

Selvadurai's configuration is axisymmetric and it is not immediately clear whether these results, in particular (3), are a consequence of this axisymmetry. Selvadurai's problem involves *complete* contact; the two half-spaces are in contact with the penny over the whole of the upper and lower faces. Gladwell and Hara (1981) had considered the *incomplete* contact problem for a rigid inclusion with faces $z = \pm f_j(r)$, where

$$f_j(r) = D_j - r^2 / (2R_j), \quad j = 1, 2. \quad (2)$$

It is interesting to note that their Fig. 1 (a sketch) shows the normal displacements equal, but not necessarily zero, where the half-spaces are in contact (for $r \geq l$). Later, in commenting on their numerical results they state "Note that the approximate w is not zero, *as it should be*, outside $r = l$ ". The italics are ours, and indeed on closer inspection we note that w need not be zero outside $r = l$. In fact there is only one easily identifiable case in which w will be zero outside $r = l$; when $\mathfrak{G}_1 = \mathfrak{G}_2, f_1(r) \equiv f_2(r)$, when the whole configuration is symmetrical about $z = 0$. However, as we will now show, the three results listed above hold for an arbitrarily shaped rigid inclusion of *constant* thickness compressed between two half-spaces.

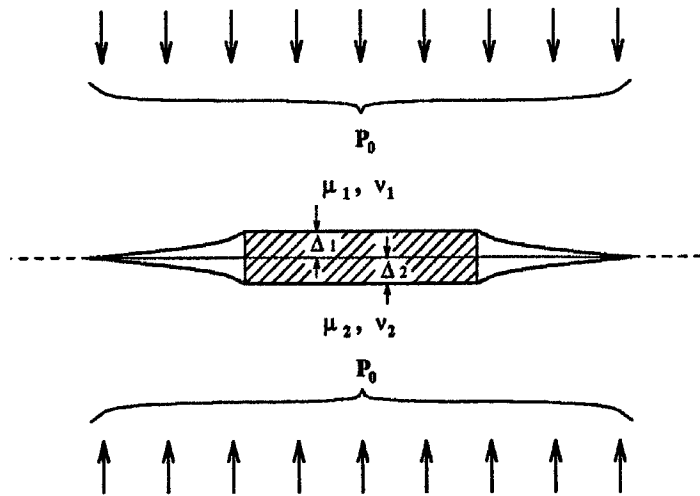


Fig. 1. A rigid disc inclusion of constant thickness compressed between two dissimilar isotropic elastic half-spaces.

2. PAPKOVICH-NEUBER SOLUTIONS

We start with the familiar Papkovitch-Neuber (P-N) representation (Gladwell, 1980)

$$2\mu\mathbf{d} = 4(1-\nu)\Psi - \nabla\{(\mathbf{r} \cdot \Psi) + \Phi\} \quad (3)$$

for the elastic displacement $\mathbf{d} \equiv (u, v, w)$ in a medium with parameters μ, ν . We suppose that there are no body forces, so that Ψ and Φ are harmonic, i.e.

$$\Delta\Psi = 0, \quad \Delta\Phi = 0. \quad (4)$$

It is well known that for many regions, of which a half-space is one, it is possible to choose Ψ so that one of its components, say Ψ_1 , is zero. This is achieved by *subtracting* from the given solution (Ψ, Φ) a *null* solution (Ψ^0, Φ^0) having $\Psi_1^0 = \Psi_1$. A null P-N solution is one for which

$$4(1-\nu)\Psi^0 - \nabla\{(\mathbf{r} \cdot \Psi^0) + \Phi^0\} \equiv 0. \quad (5)$$

It is easily shown that (Ψ^0, Φ^0) is null if

$$\Psi^0 = \nabla H \quad \text{where } \Delta H = 0 \quad (6)$$

$$\Phi = 4(1-\nu)H - \nabla(\mathbf{r} \cdot \nabla H). \quad (7)$$

Thus if Ψ_1 is to be annihilated, we must choose H , harmonic, such that

$$\frac{\partial H}{\partial z} = \Psi_1. \quad (8)$$

Barber (1992) shows how this may be done.

In our problem we are concerned with special P-N solutions, for the half-space $z \geq 0$ with the surface $z = 0$, i.e. the x, y -plane D , free of shearing stress. In such cases we customarily take

$$\Psi_1 = 0 = \Psi_2, \tag{9}$$

$$(1 - 2\nu)\Psi_3 = \frac{\partial\Phi}{\partial z}. \tag{10}$$

However, because this choice is crucial to our later argument, we first establish that it is always available, i.e. that if (Ψ, Φ) corresponds to an elastic field in $z \geq 0$, and

$$\tau_{xz} = 0 = \tau_{yz} \quad (x, y) \in D, \tag{11}$$

then we can find a null field (Ψ^0, Φ^0) such that

$$\Psi_1^0 \equiv \frac{\partial H}{\partial x} = \Psi_1, \quad \Psi_2^0 \equiv \frac{\partial H}{\partial y} = \Psi_2. \tag{12}$$

To prove this, we proceed as follows.

Equation (3) shows that the rotation component in the z -direction is

$$\omega_3 = \omega = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \frac{1}{9} \left(\frac{\partial \Psi_2}{\partial x} - \frac{\partial \Psi_1}{\partial y} \right). \tag{13}$$

It is harmonic because Ψ_1 and Ψ_2 are harmonic. The stress-strain equations give

$$\frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y} = \mu \left(\frac{\partial^2 v}{\partial x \partial z} - \frac{\partial^2 u}{\partial y \partial z} \right) = 2\mu \frac{\partial \omega}{\partial z}. \tag{14}$$

Consider the integral

$$J \equiv \int_V (\nabla \omega)^2 \, dV \tag{15}$$

over the half-space. Since ω is harmonic,

$$\operatorname{div}(\omega \nabla \omega) = \omega \Delta \omega + (\nabla \omega)^2 = (\nabla \omega)^2 \tag{16}$$

and the divergence theorem gives

$$J = - \int_D \omega \frac{\partial \omega}{\partial z} \, dS; \tag{17}$$

the necessary regularity conditions on ω at infinity are very weak. But eqns (11) and (14) show that

$$\partial \omega / \partial z = 0, \quad (x, y) \in D. \tag{18}$$

Hence $J = 0$, $\nabla \omega \equiv \mathbf{0}$ and $\omega = \text{constant} = 0$ if we take the displacement \mathbf{d} zero at infinity. Equation (13) therefore shows that any P-N solution for the half-space with D free of shear stress must satisfy

$$\frac{\partial \Psi_2}{\partial x} = \frac{\partial \Psi_1}{\partial y}. \quad (19)$$

This is clearly a *necessary* condition for the existence of an H satisfying eqn (12). Now we must find a harmonic H to satisfy eqn (12). To do this, we first construct

$$H_1 = \int_{x_s}^x \Psi_1(s, y, z) ds + \int_{y_s}^y \Psi_2(x_s, t, z) dt. \quad (20)$$

This satisfies eqn (12), but is not harmonic. However

$$\Delta H_1 = \frac{\partial}{\partial x} \Psi_1(x_s, y_s, z) + \frac{\partial}{\partial y} \Psi_2(x_s, y_s, z) = f(z), \quad (21)$$

so that we need only find $H_2 = H_2(z)$ such that

$$\Delta H_2 = H_2''(z) = -f(z). \quad (22)$$

For this we may take

$$H_2 = \int_{z_s}^z (t-z)f(t) dt \quad (23)$$

and then $H = H_1 + H_2$ is the required harmonic function.

We note that all the operations described, such as the integrations in eqns (20) and (23) are unambiguous in the half-space. We have established eqn (9); eqn (10) now follows easily from the explicit expressions for τ_{xz} , τ_{yz} in terms of Ψ_3 and Φ .

3. ANALYSIS OF THE CONTACT PROBLEM

Suppose that instead of being circular, the inclusion occupies a region D_1 in the x, y -plane, there is a separation zone D_2 , and the half-spaces are in contact over the remainder, D_3 , of the x, y -plane D .

The elastic solution for each half-space may be obtained as the superposition of two fields. The first corresponds to a uniform stress field in the z -direction, namely $\tau_{zz}^{(j)} = -p_0$, for which the normal displacements are given by

$$2(1 + \nu_j)\mu_j w_j = -p_0 z, \quad j = 1, 2. \quad (24)$$

The second may be obtained from a P-N solution

$$2\mu_j \mathbf{d}_j = 4(1 - \nu_j)\Psi^{(j)} - \nabla\{(\mathbf{r} \cdot \Psi^{(j)}) + \Phi^{(j)}\}, \quad j = 1, 2 \quad (25)$$

where $\mathbf{d}_j \equiv (u_j, v_j, w_j)$ is the vector of elastic displacements in the half-spaces (1) $z \geq 0$, (2) $z \leq 0$. Since there are no body forces present, the potentials $\Psi^{(j)}$ and $\Phi^{(j)}$ are harmonic.

There are no shearing stresses on the interface, so that

$$\tau_{xz}^{(j)} = 0 = \tau_{yz}^{(j)}, \quad x, y \in D, \quad j = 1, 2. \quad (26)$$

The analysis of section 2 shows that we may take $\Psi^{(j)} = (0, 0, \Psi_j)$ and

$$(1 - 2\nu_j)\Psi_j = \partial\Phi^{(j)}/\partial z, \quad j = 1, 2. \quad (27)$$

This yields

$$2\mu_j w_j = 2(1-\nu_j)\Psi_j - z \frac{\partial \Psi_j}{\partial z}, \quad \tau_{zz}^{(j)} = \frac{\partial \Psi_j}{\partial z} - z \frac{\partial^2 \Psi_j}{\partial z^2}, \quad j = 1, 2, \quad (28)$$

so that the boundary conditions for the Ψ_j are

$$\Psi_1 = \vartheta_1 \Delta_1, \quad \Psi_2 = -\vartheta_2 \Delta_2 \quad (x, y) \in D_1 \quad (29)$$

$$\frac{\partial \Psi_j}{\partial z} = p_0 \quad (x, y) \in D_2 \quad (30)$$

$$\frac{\Psi_1}{\vartheta_1} - \frac{\Psi_2}{\vartheta_2} = 0 \quad (x, y) \in D_3 \quad (31)$$

$$\frac{\partial \Psi_1}{\partial z} = \frac{\partial \Psi_2}{\partial z} \quad (x, y) \in D_3. \quad (32)$$

In addition there is the equilibrium condition for the inclusion, namely

$$\int_{D_1} \left(\frac{\partial \Psi_1}{\partial z} - \frac{\partial \Psi_2}{\partial z} \right) dS = 0 \quad (33)$$

and the regularity conditions at infinity, namely

$$\Psi_j(x, y, z) = O((x^2 + y^2 + z^2)^{-1/2}). \quad (34)$$

The potential Ψ_1 is harmonic in the upper half-space, and Ψ_2 in the lower half-space. Define

$$\Psi_2^*(x, y, z) = -\Psi_2(x, y, -z) \quad (35)$$

so that Ψ_2^* is harmonic in the upper half-space V . The conditions (29)–(33) translate into conditions for Ψ_1, Ψ_2^* on D , namely

$$\Psi_1 = \vartheta_1 \Delta_1, \quad \Psi_2^* = \vartheta_2 \Delta_2 \quad (x, y) \in D_1 \quad (36)$$

$$\frac{\partial \Psi_1}{\partial z} = p_0 = \frac{\partial \Psi_2^*}{\partial z} \quad (x, y) \in D_2 \quad (37)$$

$$\frac{\Psi_1}{\vartheta_1} + \frac{\Psi_2^*}{\vartheta_2} = 0 \quad (x, y) \in D_3 \quad (38)$$

$$\frac{\partial \Psi_1}{\partial z} = \frac{\partial \Psi_2^*}{\partial z} \quad (x, y) \in D_3 \quad (39)$$

$$\int_{D_1} \left(\frac{\partial \Psi_1}{\partial z} - \frac{\partial \Psi_2^*}{\partial z} \right) dS = 0. \quad (40)$$

Consider the integral

$$I \equiv \int_V \operatorname{div} \{ (\Psi_1 - \Psi_2^*) \nabla (\Psi_1 - \Psi_2^*) \} dV \quad (41)$$

taken over the upper half-space. Since $\Psi_1 - \Psi_2^*$ is harmonic in V , we have

$$I = \int_V \{\nabla(\Psi_1 - \Psi_2^*)\}^2 dV. \quad (42)$$

On the other hand, Gauss' theorem gives

$$I = - \int_D (\Psi_1 - \Psi_2^*) \frac{\partial}{\partial z} (\Psi_1 - \Psi_2^*) dS. \quad (43)$$

The conditions (37)–(39) show that there is no contribution to I from D_2 or D_3 . On D_1 both Ψ_1 and Ψ_2^* are constant so that

$$I = -(\vartheta_1 \Delta_1 - \vartheta_2 \Delta_2) \int_{D_1} \frac{\partial}{\partial z} (\Psi_1 - \Psi_2^*) dS \quad (44)$$

and this is zero because of the equilibrium condition (41). Thus $I \equiv 0$, and hence, from eqn (42), $\Psi_1 - \Psi_2^* = \text{const}$ and the regularity condition (34) shows that this must be zero. Therefore

$$\Psi_1(x, y, z) = \Psi_2^*(x, y, z) = -\Psi_2(x, y, -z) \quad (45)$$

from which the results (1)–(3) immediately follow.

4. GENERALIZATIONS

In this argument we have implicitly considered the region D_1 occupied by the inclusion to be connected, so that there is just one equilibrium condition (40). However, D_1 may be made up of a number of connected regions, so that

$$D_1 = \bigcup_{i=1}^n D_{1i} \quad (46)$$

and the inclusions may be of different thicknesses. Then the boundary conditions are

$$\Psi_1 = \vartheta_1 \Delta_{1i}, \quad \Psi_2^* = \vartheta_2 \Delta_{2i} \quad (x, y) \in D_{1i} \quad (47)$$

and the equilibrium conditions are

$$\int_{D_{1i}} \left(\frac{\partial \Psi_1}{\partial z} - \frac{\partial \Psi_2^*}{\partial z} \right) dS = 0, \quad i = 1, 2, \dots, n. \quad (48)$$

Again the argument holds and result (1) generalizes to

$$\vartheta_1 \Delta_{1i} = \vartheta_2 \Delta_{2i}, \quad i = 1, 2, \dots, n. \quad (49)$$

while (2) holds for each inclusion and (3) still holds.

Even though the results hold in this generalized configuration, a reviewer correctly commented that it is difficult to visualize a situation in which two inclusions of different thickness near to each other could maintain contact over all D_1 . For an example, we can consider two circular disc inclusions of different thicknesses which are almost in contact with each other; separation will occur over some part of the thinner inclusion.

To generalize the result further we note that the integral I in eqn (43) may be written as in eqn (44) provided that $\Psi_1 - \Psi_2^*$ is constant on D_1 ; Ψ_1 and Ψ_2^* do not have to be separately zero. Again I will be zero and eqn (45) will follow. To use this we note that if

the faces of the inclusion are plane then we may apply a rigid body rotation, of which the z -component is

$$w = \alpha x + \beta y, \quad (50)$$

to the whole space, to make $\Psi_1 - \Psi_2^*$ constant on D_1 . (We can always do this provided that D_1 is connected; if it is made up of connected regions D_{1i} , we can do it provided that the plane faces of the rigid inclusions in contact with the half-space $z > 0$ are all parallel, as are those in contact with the half-space $z < 0$.) Thus the inclusion(s) will align themselves so that, in the new frame of reference

$$\mathcal{G}_1 w_{1i} = -\mathcal{G}_2 w_{2i} \quad (x, y) \in D_{1i} \quad (51)$$

while results (2) and (3) listed in section 1 will still hold, the latter again in the new frame of reference. If the faces of the plane depart from planes, then the contact between them and the half-spaces will be incomplete and the analysis will fail.

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